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Deformations of Maxwell algebra and their dynamical realizations

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Abstract: We study all possible deformations of the Maxwell algebra. In $D = d+1 \neq 3$ dimensions there is only one-parameter deformation. The deformed algebra is isomorphic to $so(d+1,1) \oplus so(d,1)$ or to $so(d,2) \oplus so(d,1)$ depending on the signs of the deformation parameter. We construct in the $dS(AdS)$ space a model of massive particle interacting with Abelian vector field via non-local Lorentz force. In $D = 2+1$ the deformations depend on two parameters b and k. We construct a phase diagram, with two parts of the (b,k) plane with $so(3,1) \oplus so(2,1)$ and $so(2,2) \oplus so(2,1)$ algebras separated by a critical curve along which the algebra is isomorphic to $Iso(2,1) \oplus so(2,1)$. We introduce in $D = 2+1$ the Volkov-Akulov type model for a Abelian Goldstone-Nambu vector field described by a non-linear action containing as its bilinear term the free Chern-Simons Lagrangean.

Keywords: Space-Time Symmetries, Field Theories in Lower Dimensions, Spontaneous Symmetry Breaking, Global Symmetries

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Contents

1 Introduction

It is known since 1970, see [1], that the presence of a constant classical EM field background in Minkowski space-time leads to the modification of Poincare symmetries. One obtains the enlargement of Poincare algebra, called Maxwell algebra [2, 3] which is obtained by the replacement of the commutative momentum generators $P_a, (a = 0, 1, \ldots, d)$ by

$$[P_a,\ P_b] = i\ eZ_{ab}, \qquad Z_{ba} = -Z_{ab}, \tag{1.1}$$

where e is the electromagnetic coupling constant.

It is known that the Poincare algebra does not permit any central extensions in $D = d+1\,(d>1)$ dimensions, see for example [4]. The new generators Z_{ab} describe so called tensorial central charges[1] and satisfy the relations

$$\begin{aligned}
&[M_{ab},\ Z_{cd}] = -i\,(\eta_{bc}\ Z_{ad} - \eta_{bd}\ Z_{ac} + \eta_{ad}\ Z_{bc} - \eta_{ac}\ Z_{bd}),\\
&[P_a,\ Z_{bc}] = 0, \qquad [Z_{ab},\ Z_{cd}] = 0.
\end{aligned} \tag{1.2}$$

A dynamical realization of Maxwell algebra can be obtained by considering the relativistic particle coupled in minimal way to the electromagnetic potential $A_b = \frac{1}{2} f^0_{ab} x^a$ defining the constant field strength $F_{ab} = f^0_{ab}$. The respective first order lagrangian is the following

$$L = \pi_a \dot{x}^a - \frac{\lambda}{2}(\pi^2 + m^2) + \frac{e}{2} f^0_{ab} x^a \dot{x}^b. \tag{1.3}$$

The coordinate π_a can be expressed in terms of the canonical momenta p_a conjugated to x^a as

$$\pi_a = p_a + \frac{e}{2} f^0_{ab} x^b. \tag{1.4}$$

From (1.3) we get the second order lagrangian

$$L = -m\sqrt{-\dot{x}^2} + \frac{e}{2} f^0_{ab} x^a \dot{x}^b. \tag{1.5}$$

Note that this action is not invariant under the whole Maxwell algebra since part of the Lorentz rotations is broken by the choice of constant electromagnetic field f^0_{ab}. In order to recover the Maxwell symmetry one has to promote f^0_{ab} to be the dynamical degrees of freedom and consider an extension of space-time by supplementing the new coordinates $\theta^{ab}(= -\theta^{ba})$ which are canonically conjugated to Z_{ab}. In order to introduce the dynamics invariant under the Maxwell group symmetries we have applied in [13, 14] the method of non-linear realizations employing the Maurer Cartan (MC) one-forms (see e.g. [15, 16]).

The aim of this paper is to study all possible deformations of the Maxwell algebra (1.1), (1.2), and investigate the dynamics realizing the deformed Maxwell symmetries.

In $D{\neq}2{+}1$ there exists only one-parameter deformation which leads for positive (negative) value of the deformation parameter k to an algebra that is isomorphic to the direct sum of the AdS algebra $so(d,2)$ (dS algebra $so(d+1,1)$) and the Lorentz algebra $so(d,1)$. We stress here that this deformation for $k>0$ has been firstly obtained by Soroka and Soroka who called the Maxwell algebra as the tensor extension of Poincare algebra [17, 18].

In $D{=}2{+}1$ one gets a two-parameter family of deformations, with second deformation parameter b. The parameter space (b,k) is divided in two regions separated by the critical curve

$$A(b,k) = \left(\frac{k}{3}\right)^3 - \left(\frac{b}{2}\right)^2 = 0 \tag{1.6}$$

[1] In euclidean spaces R^{2n} and R^{4n} with automorphism groups $\mathrm{U}(n)$ and $\mathrm{Sp}(n)\times\mathrm{SU}(2)$ (Kähler and hyper-Kähler geometries), Galperin et.al. [5, 6] have obtained scalar as well as tensorial central extensions (a triplet in hyper-Kähler). In the literature the tensorial central charges were introduced mostly in the Poincare superalgebras [7–9] and also in p-brane non-relativistic Galilei and Newton-Hooke algebras [10–12].

on which the deformed algebra is non-semisimple. It appears that for $A > 0$ ($A < 0$) the deformed algebra is isomorphic to $so(2,2) \oplus so(2,1)$ ($so(3,1) \oplus so(2,1)$). On the curve (1.6) the deformed algebra is the direct sum of D=2+1 Poincare algebra and D=2+1 Lorentz algebra, $Iso(2,1) \oplus so(2,1)$.

In order to study the particle dynamics in the deformed cases we consider the MC one-forms on the suitable coset of deformed Maxwell group. Firstly we obtain, for arbitrary D and $k \neq 0, b = 0$, the particle model in curved and enlarged space-time $y^A = (x^a, \theta^{ab})$. We choose the coset which leads to the metric depending only on the space-time coordinates x^a. We derive in such a case the particle model in AdS (for $k > 0$) or dS (for $k < 0$) curved space-time with the coupling to Abelian vector field which generalizes, in the theory with deformed Maxwell symmetry, the Lorentz force term describing the particle interaction with constant electromagnetic field. The Lorentz force in the case studied here becomes non-local.

In D=2+1 and $k = 0, b \neq 0$, we will consider a nonlinear field theory realization of the deformed Maxwell algebra in six-dimensional enlarged space ($x^a, \theta^a = \frac{1}{2}\epsilon^{abc}\theta_{bc}$; $a, b = 0, 1, 2$) by assuming that the surface $\theta^a = \theta^a(x)$ describes D=2+1 dimensional Goldstone vector fields.[2] If we postulate the action of Volkov-Akulov type [19, 20] we shall obtain the field theory in D=2+1 space-time with a lagrangian containing a free Abelian Chern-Simons term [21–23].

The organization of the paper is as follows. In section 2 we review some properties of the Maxwell group and consider the corresponding particle model. In section 3 we will present all possible deformations of Maxwell algebra. In section 4 we construct the deformed particle model for arbitrary D with $k \neq 0, b = 0$. In section 5 we consider D=2+1 case with $k = 0, b \neq 0$ and promote the group parameters θ^a to Goldstone fields $\theta^a(x)$. These Goldstone-Nambu fields will be described by Volkov-Akulov type action. In the final section we present a short summary and further outlook. Some technical details are added in two appendices.

2 Particle model from the Maxwell algebra

In this section we construct a particle model invariant under the complete Maxwell algebra. Such a model can be derived geometrically [14] by the techniques of non-linear realizations, see e.g. [15], and by the introduction of new dynamical coordinates f_{ab} that transform covariantly under the Maxwell group.

Let us consider the coset [13, 14]

$$g = e^{iP_a x^a} e^{\frac{i}{2} Z_{ab}\theta^{ab}}. \tag{2.1}$$

The corresponding Maurer-Cartan (MC) one-forms are

$$\Omega = -ig^{-1}dg = P_a\ e^a + \frac{1}{2} Z_{ab}\ \omega^{ab} + \frac{1}{2} M_{ab}\ l^{ab}, \tag{2.2}$$

[2]Such a method was used firstly by Volkov and Akulov to derive the Goldstino field action [19].

where

$$e^a = dx^a, \qquad \omega^{ab} = d\theta^{ab} + \frac{1}{2}(x^a\, dx^b - x^b\, dx^a), \qquad l^{ab} = 0. \tag{2.3}$$

Differential realization of the Maxwell algebra generators is described by the left invariant vector fields in the extended space-time $y^A = (x^a, \theta^{ab})$,[3] which are dual to the one-forms (2.3) [14, 17].

A first order form of the lagrangian for the particle invariant under the full Maxwell algebra with the coordinates f_{ab} describing new dynamical coupling can be written as [13][4]

$$\tilde{L} = \pi_a e^a{}_A \dot{y}^A + \frac{1}{2} f_{ab}\, \omega^{ab}{}_A \dot{y}^A - \frac{\lambda}{2}(\pi^2 + m^2), \tag{2.4}$$

where

$$e^a = e^a{}_A\; dy^A, \qquad \omega^{ab} = \omega^{ab}{}_A\; dy^A, \tag{2.5}$$

more explicitly,

$$\begin{aligned} e^a{}_b &= \delta^a{}_b, & e^a{}_{bc} &= 0, \\ \omega^{ab}{}_c &= \frac{1}{2}(x^a \delta^b{}_c - x^b \delta^a{}_c), & \omega^{ab}{}_{cd} &= \frac{1}{2}(\delta^a{}_c \delta^b{}_d - \delta^a{}_d \delta^b{}_c). \end{aligned} \tag{2.6}$$

From the (2.4) we obtain the second order lagrangian

$$\mathcal{L} = -m\sqrt{-\dot{x}^2} + \frac{1}{2} f_{ab} \left(\dot{\theta}^{ab} + \frac{1}{2}(x^a \dot{x}^b - x^b \dot{x}^a) \right) = -m\sqrt{-\dot{x}^2} + \hat{A}^*. \tag{2.7}$$

The Euler-Lagrange equations of motion are

$$\dot{f}_{ab} = 0, \tag{2.8}$$

$$\dot{\theta}^{ab} = -\frac{1}{2}(x^a \dot{x}^b - x^b \dot{x}^a), \tag{2.9}$$

$$m\ddot{x}_a = f_{ab}\dot{x}^b, \tag{2.10}$$

where we took a proper time gauge in (2.10). Integration of (2.8) gives $f_{ab} = f^0_{ab}$ and such a solution breaks the Lorentz symmetry spontaneously into a subalgebra of Maxwell algebra. Substituting this solution in the equation of motion (2.10) we provide the motion of a particle in the constant electromagnetic field [1, 2] described by the lagrangian (1.5). From (2.9) one can conjecture that the new coordinates θ^{ab} are related with the angular momenta.

Notice that the interaction part of the lagrangian (2.7) defines an analogue of the EM potential $\hat{A}$ as one-form in the extended bosonic space (x, θ, f)

$$\hat{A} = \frac{1}{2}\, f_{ab}\, \omega^{ab} \tag{2.11}$$

The closed two form

$$\hat{F} = d\hat{A} = \frac{1}{2}\, f_{ab}\, e^a \wedge e^b + \frac{1}{2}\, df_{ab} \wedge \omega^{ab} \tag{2.12}$$

is such that the second term vanishes on shell (2.8). We see that on-shell the field strength has the constant components f_{ab}.

[3] We could also have a realization in terms of right invariant vector fields that generate the infinitesimal transformations (2.13).

[4] In the following we put the electric charge e equal to 1 for simplicity.

2.1 Phase space realization and Casimir operators

The infinitesimal symmetries of the lagrangian (2.7) are[5]

$$
\begin{array}{llll}
P_a: & \delta x^a = \epsilon^a, & \delta\theta^{ab} = -\frac{1}{2}(\epsilon^a x^b - \epsilon^b x^a), & \\
M_{ab}: & \delta x^a = \lambda^a{}_b\, x^b, & \delta\theta^{ab} = \lambda^{[a}{}_c\, \theta^{cb]}, & \delta f^{ab} = \lambda^{[a}{}_c\, f^{cb]}, \quad \lambda^{ab} + \lambda^{ba} = 0, \\
Z_{ab}: & \delta\theta^{ab} = \epsilon^{ab}, & \epsilon^{ab} + \epsilon^{ba} = 0. &
\end{array}
\tag{2.13}
$$

The corresponding Noether canonical generators are

$$
\begin{aligned}
\mathcal{P}_a &= -\left(p_a - \frac{1}{2}\, p_{ab} x^b\right), \\
\mathcal{M}_{ab} &= -\left(p_{[a} x_{b]} + p_{[ac}\theta_{b]}{}^c + p^f_{[ac} f_{b]}{}^c\right), \\
\mathcal{Z}_{ab} &= -\, p_{ab}.
\end{aligned}
\tag{2.14}
$$

They realize the Maxwell algebra (1.1) and (1.2), where p_a, p_{ab}, p_f^{ab} are the canonically conjugated momenta of the coordinates x^a, θ^{ab}, f_{ab}.

From the lagrangian (2.7) we obtain the constraints

$$
\begin{aligned}
\phi &= \frac{1}{2}\left(\pi_a^2 + m^2\right) = 0, \qquad\qquad \pi_a \equiv p_a + \frac{1}{2}\, f_{ab}\, x^b, \\
\phi_{ab} &= p_{ab} - f_{ab} = 0, \\
\phi_f^{ab} &= p_f^{ab} = 0.
\end{aligned}
\tag{2.15}
$$

The last two are the second class constraints and are solved by the choice $(f_{ab}, p_f^{ab}) = (p_{ab}, 0)$.

The Hamiltonian becomes

$$
\mathcal{H} = \lambda\, \phi = \frac{\lambda}{2}\left(\pi_a^2 + m^2\right)
\tag{2.16}
$$

and the Hamilton equations are, using $p_{ab} = f_{ab}$,

$$
\begin{aligned}
\dot{x}^a &= \lambda\, \pi^a, & \dot{p}_a &= \frac{\lambda}{2}\, f_{ac}\, \pi^c, \\
\dot{\theta}^{ab} &= \frac{\lambda}{2}\, \pi^{[a}\, x^{b]}, & \dot{f}_{ab} &= 0.
\end{aligned}
\tag{2.17}
$$

It follows

$$
\dot{\pi}_a = \lambda\, f_{ac}\, \pi^c\,,
\tag{2.18}
$$

then the constraints (2.15) and the global generators (2.14) are conserved.

There are four Casimirs in the Maxwell algebra in four dimensions, [2, 17]

$$
\begin{aligned}
C_1 &= \mathcal{P}^2 - \mathcal{M}_{ab}\mathcal{Z}^{ab}, & C_2 &= \frac{1}{2}\,\mathcal{Z}^2, \\
C_3 &= (\mathcal{Z}\tilde{\mathcal{Z}}), & C_4 &= (\mathcal{P}^b\, \tilde{\mathcal{Z}}_{ba})^2 + \frac{1}{4}(\mathcal{Z}\tilde{\mathcal{Z}})\,(\mathcal{M}_{ab}\, \tilde{\mathcal{Z}}^{ab}),
\end{aligned}
\tag{2.19}
$$

[5] Our convention of anti-symmetrization is $A_{[a}B_{b]} = A_a B_b - A_b B_a$.

where $\tilde{\mathcal{Z}}^{ab} = \frac{1}{2}\epsilon^{abcd}\mathcal{Z}_{cd}$. The values of the Casimirs are, using the standard $D = 4$ notation ($\mathbf{B} = f^{ij}, \mathbf{E} = f^{0i}$),

$$\begin{aligned} C_1 &= \pi^2 = -m^2, & C_2 &= \frac{1}{2}\,f^2 = \mathbf{B^2} - \mathbf{E^2}, \\ C_3 &= \frac{1}{2}\,\epsilon^{abcd} f_{ab} f_{cd} = 4\,\mathbf{B}\cdot\mathbf{E}, & C_4 &= \frac{1}{2}\,m^2\,f^2 + (\pi_b\,f^{ba})^2 = m^2(\mathbf{B^2} + \mathbf{E^2}). \end{aligned} \tag{2.20}$$

where in the second term of C_4 we took a frame in which $\pi_a = (m, 0, 0, 0)$ and imposed the mass shell constraint. In more general case of time-like π_a, $\mathbf{B}, \mathbf{E}$ are defined relatively to the direction of π_a, so that expressions for C's remain the ones given by the formulae (2.20).

2.2 First-quantized theory

Let us observe from (2.15) that the equation $\phi = C_1 + m^2 = 0$ represents unique first class constraint in the model. If we introduce first-quantized theory, in the Schrödinger representation, we obtain the following generalized KG equation,

$$\left[\left(\frac{1}{i}\frac{\partial}{\partial x^a} + \frac{1}{2i}x^b\frac{\partial}{\partial\theta^{ab}}\right)^2 + m^2\right]\Psi(x^a, \theta^{ab}) = 0. \tag{2.21}$$

In general case the remaining three Casimirs C_2, C_3, C_4 are not restricted, however in order to get the irreducible representation it is necessary to impose their definite values by three differential equations

$$C_j\,\Psi(x^a, \theta^{ab}) = \lambda_j\,\Psi(x^a, \theta^{ab}), \qquad (j = 2, 3, 4), \tag{2.22}$$

where

$$\begin{aligned} C_2 &= -\frac{1}{2}\,\frac{\partial}{\partial\theta^{ab}}\frac{\partial}{\partial\theta_{ab}}, \qquad C_3 = -\frac{1}{2}\,\epsilon^{abcd}\frac{\partial}{\partial\theta^{ab}}\frac{\partial}{\partial\theta^{cd}}, \\ C_4 &= -\frac{m^2}{2}\,\frac{\partial}{\partial\theta^{ab}}\frac{\partial}{\partial\theta_{ab}} + \left(\left(\frac{\partial}{\partial x^b} + \frac{1}{2}x^c\frac{\partial}{\partial\theta^{bc}}\right)\frac{\partial}{\partial\theta_{ba}}\right)^2. \end{aligned} \tag{2.23}$$

The constraints $C_j = \lambda_j$ can be incorporated into our particle model by introducing suitable lagrangian multipliers in (2.4).

3 Deformations of Maxwell algebra

3.1 General considerations

In this section we would like to find all possible deformations of the Maxwell algebra. The problem of finding the continuous deformations of a Lie algebra can be described in cohomological terms [24]. We first consider the Lie algebra-valued Maurer-Cartan (MC) one-form

$$\Omega = -i\,g^{-1}dg = \lambda^a G_a, \qquad [G_a, G_b] = i\,C^c_{ab}\,G_c, \tag{3.1}$$

where G_a's are generators of a Lie algebra with the structure constants C^c_{ab} and λ^a is the basis of left-invariant one-forms. The MC equation, $d\Omega + i\Omega\wedge\Omega = 0$, becomes

$$d\lambda^a = -\frac{1}{2}C_b{}^a{}_c\,\lambda^b\wedge\lambda^c, \tag{3.2}$$

and describes the Lie algebra in terms of dual forms.

We define the matrix-valued one-form $C^a{}_b = \lambda^c C_c{}^a{}_b$ and following the notation of [25] we consider the covariant exterior differential $D \equiv d + C\wedge$ with $D\wedge D = 0$. The infinitesimal deformations are characterized by the non-trivial vector-valued two-forms $A^{(2)}$ verifying

$$DA^{(2)} = 0\,, \qquad A^{(2)} \neq -D\Phi^{(1)}. \tag{3.3}$$

Therefore the non-trivial infinitesimal deformations are in one to one correspondence with the second cohomology group $H^2(g;g)$. If a non-trivial linear (infinitesimal) deformation $A^{(2)}$ is found, the next step is to investigate the Jacobi identities of higher order in the deformations parameters. The quadratic and higher deformations are controlled by the cohomology $H^3(g;g)$. In the case when $H^3(g;g)$ vanishes, it is always possible to choose a representative in the class of infinitesimal deformations such that it verifies the Jacobi identity in all orders.

Let us apply the above ideas to the Maxwell algebra (1.1). The MC form for the Maxwell algebra is

$$\Omega = P_a L_P^a + \frac{1}{2} Z_{ab} L_Z^{ab} + \frac{1}{2} M_{ab} L_M^{ab} \tag{3.4}$$

The MC equations in this case are given by[6]

$$\begin{aligned} dL_M^{ab} + L_M^{ac} L_{Mc}{}^b &= 0, \\ dL_P^a + L_M^{ac} L_{Pc} &= 0, \\ dL_Z^{ab} + L_Z^{ac} L_{Mc}{}^b + L_M^{ac} L_{Zc}{}^b - L_P^a L_P^b &= 0. \end{aligned} \tag{3.5}$$

Expanding the vector-valued two-form $A^{(2)}$ on the basis of one-forms L's and solving the linear equations resulting from (3.3) we find a one-parameter family of non-trivial solutions for $A^{(2)}$, with the exception that there is a two-parameter family in "exotic" case $D{=}2{+}1$.[7] Infinitesimal deformations found in this way are not unique but have an ambiguity described by $D\Phi^{(1)}$. Since $H^3(g;g)$ vanishes[8] finite deformations are found by adjusting the trivial one-form in a way providing the Jacobi identities for finite values of the deformation parameters. We find that for any dimension D there is a one-parameter family of exact Lie algebras, but for $D{=}2{+}1$ there exists a two-parameter family. The MC equations get additional terms representing deformations as follows

$$\begin{aligned} dL_M^{ab} + L_M^{ac} L_{Mc}{}^b &= b\,\epsilon^{abc} L_{Zcd} L_P^d, \\ dL_P^a + L_M^{ac} L_{Pc} &= k\, L_Z^{ac} L_{Pc} + b\,\frac{1}{4} L_Z^{ab} \epsilon_{bcd} L_Z^{cd}, \\ dL_Z^{ab} + L_Z^{ac} L_{Mc}{}^b + L_M^{ac} L_{Zc}{}^b - L_P^a L_P^b &= k\, L_Z^{ac} L_{Zc}{}^b, \qquad (\epsilon^{012} = -\epsilon_{012} = 1). \end{aligned} \tag{3.6}$$

Here k and b are arbitrary real constant parameters; we stress that deformation terms proportional to b are present only in $D = 2{+}1$. The length dimensions of k and b are respectively $[L^{-2}]$ and $[L^{-3}]$. In next two subsections we will study these continuous deformations using the Lie algebra generators.

[6] As usual we will often omit "$\wedge$" for exterior product of forms.

[7] Some of the calculations with forms are being done using the Mathematica code for differential forms EDC [26].

[8] We acknowledge Sotirios Bonanos for discussions on this point

3.2 Arbitrary dimensions

The general deformed Maxwell algebra found in the previous subsection can be written in terms of the commutators of generators. In general dimensions there exists only the following k-deformed algebra, with $b = 0$

$$\begin{aligned}
[P_a,\ P_b] &= i\ Z_{ab}, & [M_{ab},\ M_{cd}] &= -i\ \eta_{b[c}\ M_{ad]} + i\ \eta_{a[c}\ M_{bd]}, \\
[P_a,\ M_{bc}] &= -i\ \eta_{a[b}\ P_{c]}, & [Z_{ab},\ M_{cd}] &= -i\left(\ \eta_{b[c}\ Z_{ad]} -\ \eta_{a[c}\ Z_{bd]}\right), \\
[P_a,\ Z_{bc}] &= +ik\,\eta_{a[b}\ P_{c]}, \\
[Z_{ab},\ Z_{cd}] &= +ik\left(\ \eta_{b[c}\ Z_{ad]} - i\ \eta_{a[c}\ Z_{bd]}\right). & & \qquad (3.7)
\end{aligned}$$

For $k \neq 0$ case we introduce dimensionless rescaled generators as

$$\mathcal{P}_a = \frac{P_a}{\sqrt{|k|}}, \qquad \mathcal{M}_{ab} = -\frac{Z_{ab}}{k}, \qquad \mathcal{J}_{ab} = M_{ab} + \frac{Z_{ab}}{k}, \qquad (3.8)$$

then the k-deformation of Maxwell algebra becomes

$$\begin{aligned}
[\mathcal{P}_a,\ \mathcal{P}_b] &= -i\,\frac{k}{|k|}\ \mathcal{M}_{ab}, \\
[\mathcal{P}_a,\ \mathcal{M}_{bc}] &= -i\ \eta_{a[b}\ \mathcal{P}_{c]}, & [\mathcal{M}_{ab},\ \mathcal{M}_{cd}] &= -i\ \eta_{b[c}\ \mathcal{M}_{ad]} + i\ \eta_{a[c}\ \mathcal{M}_{bd]}, \\
[\mathcal{P}_a,\ \mathcal{J}_{bc}] &= [\mathcal{M}_{ab},\ \mathcal{J}_{cd}] = 0, & [\mathcal{J}_{ab},\ \mathcal{J}_{cd}] &= -i\ \eta_{b[c}\ \mathcal{J}_{ad]} + i\ \eta_{a[c}\ \mathcal{J}_{bd]}. \qquad (3.9)
\end{aligned}$$

The algebra of $(\mathcal{P}_a,\ \mathcal{M}_{cd},\ \mathcal{J}_{cd},)$ for $k > 0$ (k^+-deformation) is $so(D-1,2) \oplus so(D-1,1)$, i.e. we obtain the direct sum of AdS_D and D-dimensional Lorentz group. For $k < 0$ (k^--deformation) we get $so(D,1) \oplus so(D-1,1)$, i.e., the direct sum of dS_D and D-dimensional Lorentz group. We recall here that the above algebra for $k > 0$ was previously found by Soroka and Soroka [18]. In our further discussion we will also use the notation $k = \pm\frac{1}{R^2}$ where R is the radius of AdS ($k > 0$) or dS ($k < 0$) space.

3.3 $D = 2 + 1$

This case is interesting since there is an exotic b-deformation of the Maxwell algebra in addition to the k-deformation. In $D = 2+1$ it is convenient to use the dual vectors for anti-symmetric tensors, i.e.

$$M^a = \frac{1}{2}\epsilon^{abc} M_{bc}, \qquad Z^a = \frac{1}{2}\epsilon^{abc} Z_{bc}, \qquad etc. \qquad (3.10)$$

The algebra (3.7) looks as follows

$$\begin{aligned}
[P_a, P_b] &= -i\epsilon_{abc} Z^c, \quad [M_a, M_b] = i\epsilon_{abc} M^c, \\
[P_a, M_b] &= i\epsilon_{abc} P^c, \qquad [Z_a, M_b] = i\epsilon_{abc} Z^c, \\
[P_a, Z_b] &= -ik\,\epsilon_{abc} P^c - ib\,\epsilon_{abc} M^c, \\
[Z_a, Z_b] &= -ik\,\epsilon_{abc} Z^c + i\,b\,\epsilon_{abc} P^c. \qquad (3.11)
\end{aligned}$$

For $b=0,\ k\neq 0$, as was discussed previously, the algebra is $so(2,2)\oplus so(2,1)$ for $k>0$ (k^+-deformation) and $so(3,1)\oplus so(2,1)$ for $k<0$ (k^--deformation). We rewrite the formula (3.8) in a matrix form as

$$\begin{pmatrix}\mathcal{P}_a\\ \mathcal{M}_a\\ \mathcal{J}_a\end{pmatrix}=\begin{pmatrix}\ \ U_k\ \ \end{pmatrix}\begin{pmatrix}P_a\\ M_a\\ Z_a\end{pmatrix},\qquad U_k=\begin{pmatrix}\frac{1}{\sqrt{|k|}} & 0 & 0\\ 0 & 0 & -\frac{1}{k}\\ 0 & 1 & \frac{1}{k}\end{pmatrix}. \tag{3.12}$$

The algebra (3.9) becomes

$$\begin{aligned}&[\mathcal{P}_a,\mathcal{P}_b]=i\,\frac{k}{|k|}\,\epsilon_{abc}\mathcal{M}^c,\quad [\mathcal{P}_a,\mathcal{M}_b]=i\,\epsilon_{abc}\mathcal{P}^c,\quad [\mathcal{M}_a,\mathcal{M}_b]=i\,\epsilon_{abc}\mathcal{M}^c,\\ &[\mathcal{J}_a,\mathcal{J}_b]=i\epsilon_{abc}\mathcal{J}^c,\qquad\qquad [\mathcal{P}_a,\mathcal{J}_b]=[\mathcal{M}_a,\mathcal{J}_b]=0.\end{aligned} \tag{3.13}$$

For $k=0,\ b\neq 0$ (b-deformation) we can introduce

$$\begin{pmatrix}\mathcal{P}_a\\ \mathcal{M}_a\\ \mathcal{J}_a\end{pmatrix}=\begin{pmatrix}\ \ U_b\ \ \end{pmatrix}\begin{pmatrix}P_a\\ M_a\\ Z_a\end{pmatrix},\qquad U_b=\begin{pmatrix}\frac{1}{\sqrt{3}\,b^{1/3}} & 0 & \frac{1}{\sqrt{3}\,b^{2/3}}\\ -\frac{1}{3\,b^{1/3}} & \frac{2}{3} & \frac{1}{3\,b^{2/3}}\\ \frac{1}{3\,b^{1/3}} & \frac{1}{3} & -\frac{1}{3\,b^{2/3}}\end{pmatrix} \tag{3.14}$$

and show that

$$\begin{aligned}&[\mathcal{P}_a,\mathcal{P}_b]=-i\,\epsilon_{abc}\mathcal{M}^c,\quad [\mathcal{P}_a,\mathcal{M}_b]=i\epsilon_{abc}\mathcal{P}^c,\quad [\mathcal{M}_a,\mathcal{M}_b]=i\epsilon_{abc}\mathcal{M}^c,\\ &[\mathcal{J}_a,\mathcal{J}_b]=i\epsilon_{abc}\mathcal{J}^c,\qquad [\mathcal{P}_a,\mathcal{J}_b]=[\mathcal{M}_a,\mathcal{J}_b]=0.\end{aligned} \tag{3.15}$$

Then $(\mathcal{P}_a,\mathcal{M}_a)$ are the $so(3,1)$ generators and $\mathcal{J}_a$ describes $so(2,1)$. This algebra is isomorphic to the one with $b=0,k<0$ (k^--deformation) (3.13).

To examine more general case with any values of the deformation parameters (b,k) we consider the Killing form of the algebra (3.11),

$$g_{ij}=C_{ik}^{\ell}C_{\ell j}^{k}=6\begin{pmatrix}1 & 0 & -\frac{2k}{3}\\ 0 & \frac{2k}{3} & -b\\ -\frac{2k}{3} & -b & \frac{2k^2}{3}\end{pmatrix}\otimes\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}. \tag{3.16}$$

where C_{ij}^k is the structure constant in the base of nine generators $G_i=(P_a,M_a,Z_a)$. Its determinant is

$$\det g_{ij}=6^9 4^3\,A(b,k)^3,\qquad A(b,k)\equiv\left(\frac{k}{3}\right)^3-\left(\frac{b}{2}\right)^2. \tag{3.17}$$

In the case $\det g=0$ the Killing form is degenerate, otherwise the algebra is semisimple.

In figure 1 we illustrate the parameter plane (b,k) which is divided into four regions in the table 1; The origin, $b=k=0$ on the figure (see I) is the case of original Maxwell algebra. When (b,k) belongs to one of the two branches ($b>0,k>0$ and $b<0,k>0$) of the degenerate curve (see II) we find that the algebra is a direct sum of D=2+1 Poincare ($Iso(2,1)$) and $so(2,1)$. The generators are

$$\begin{pmatrix}\mathcal{P}_a\\ \mathcal{M}_a\\ \mathcal{J}_a\end{pmatrix}=\begin{pmatrix}(\frac{2}{b})^{1/3} & 2 & (\frac{2}{b})^{2/3}\\ -\frac{2}{9}\,(\frac{2}{b})^{1/3} & \frac{8}{9} & \frac{1}{9}\,(\frac{2}{b})^{2/3}\\ \frac{2}{9}\,(\frac{2}{b})^{1/3} & \frac{1}{9} & -\frac{1}{9}\,(\frac{2}{b})^{2/3}\end{pmatrix}\begin{pmatrix}P_a\\ M_a\\ Z_a\end{pmatrix} \tag{3.18}$$

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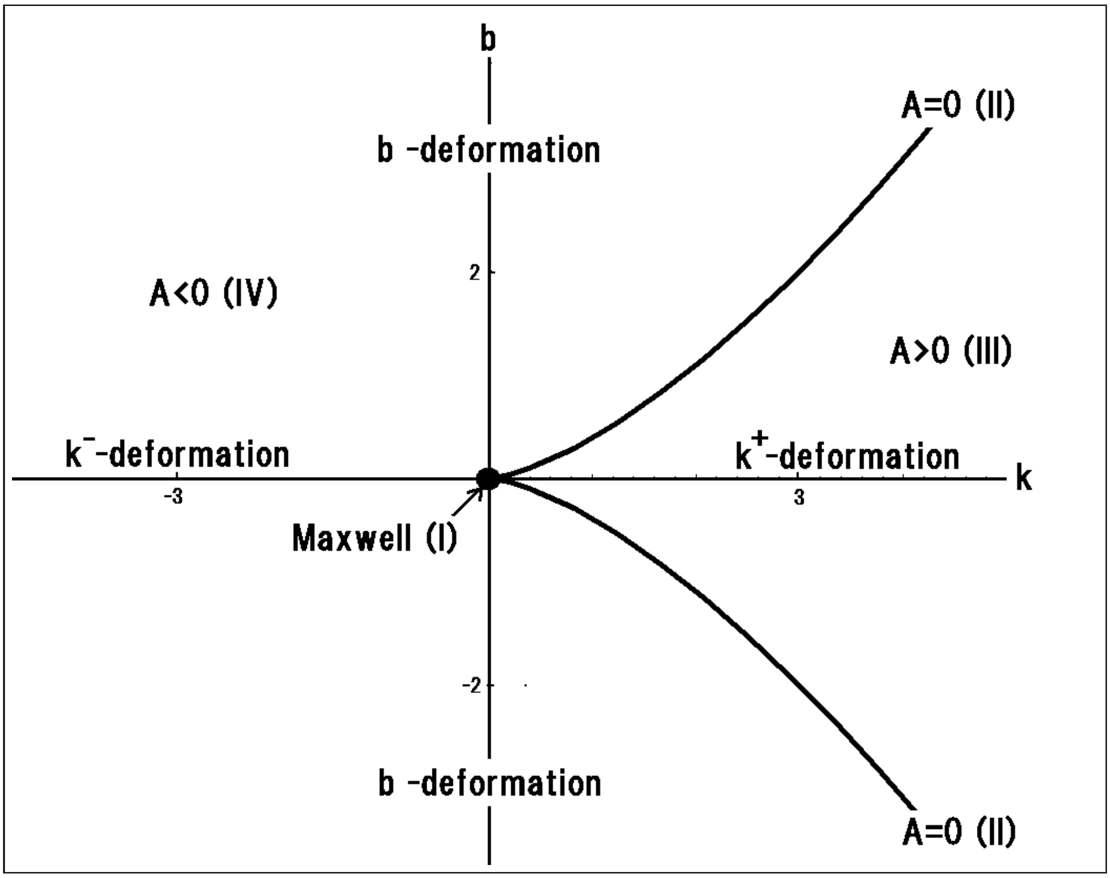

Figure 1. The phase diagram for deformed D=2+1 Maxwell algebra.

I	$\det g = 0$	$b = 0,\ k = 0,$	Maxwell	Maxwell algebra
II	$\det g = 0$	$A(b,k) = 0$	Poincaré	$Iso(2,1) \oplus so(2,1)$
III	$\det g > 0$	$A(b,k) > 0$	AdS	$so(2,2) \oplus so(2,1)$
IV	$\det g < 0$	$A(b,k) < 0$	dS	$so(3,1) \oplus so(2,1)$

Table 1. The phase sectors for deformed D=2+1 Maxwell algebra.

and satisfy

$$[\mathcal{P}_a, \mathcal{P}_b] = 0, \qquad [\mathcal{P}_a, \mathcal{M}_b] = i\epsilon_{abc}\mathcal{P}^c, \qquad [\mathcal{M}_a, \mathcal{M}_b] = i\epsilon_{abc}\mathcal{M}^c,$$
$$[\mathcal{J}_a, \mathcal{J}_b] = i\epsilon_{abc}\mathcal{J}^c, \quad [\mathcal{P}_a, \mathcal{J}_b] = [\mathcal{M}_a, \mathcal{J}_b] = 0. \tag{3.19}$$

The AdS region (see III) includes k^+-deformation for which the algebra is $so(2,2) \oplus so(2,1)$. One can show that the deformed algebra for any internal point (b,k) in sector (III) is isomorphic to $so(2,2) \oplus so(2,1)$. The generators are constructed as linear combinations of (P, M, Z),

$$\begin{pmatrix} \mathcal{P}_a \\ \mathcal{M}_a \\ \mathcal{J}_a \end{pmatrix} = \begin{pmatrix} \\ U^{+}(b,k) \\ \\ \end{pmatrix} \begin{pmatrix} P_a \\ M_a \\ Z_a \end{pmatrix}, \qquad U^{+}(b,k) \in GL(3,R) \tag{3.20}$$

and $(\mathcal{P}_a, \mathcal{M}_a, \mathcal{J}_a)$ verify the same AdS algebra as (3.13) with $k > 0$. The explicit form of the matrix $U^{+}(b,k)$ is discussed in the appendix. It is ill-defined as (b,k) approaches the boundary (II) because of the appearance of singular coefficients.

The dS region (see IV) includes k^{-}- and b-deformations for which the algebra is isomorphic to $so(3,1) \oplus so(2,1)$. It is true for general deformations corresponding to any internal point (b,k) in sector (IV),

$$\begin{pmatrix} \mathcal{P}_a \\ \mathcal{M}_a \\ \mathcal{J}_a \end{pmatrix} = \begin{pmatrix} \\ U^{-}(b,k) \\ \\ \end{pmatrix} \begin{pmatrix} P_a \\ M_a \\ Z_a \end{pmatrix}, \qquad U^{-}(b,k) \in GL(3,R), \tag{3.21}$$

and $(\mathcal{P}_a, \mathcal{M}_a, \mathcal{J}_a)$ verify the same dS algebra as (3.13) with $k < 0$, or equivalently given by (3.15). It is shown in appendix A that when $k = 0$ the transformation matrix $U^{-}(b,k)$ is U_b described by (3.14), while it becomes U_k of k^{-}-deformation (3.12) when $b = 0, k < 0$. It is singular as (b,k) approaches the boundary (II), similar as in region (III).

Finally we observe that if a new length scale R' is introduced by the relation

$$b = \frac{1}{R'^3} \tag{3.22}$$

the critical curve equation $A = 0$ (see (1.6)) is described by two half-lines relating the parameters R and R',

$$R' = \pm \left(\frac{3^{\frac{1}{2}}}{2^{\frac{1}{3}}} \right) R, \quad (R > 0). \tag{3.23}$$

4 Particle models on the k-deformed Maxwell algebra

In this section we will discuss a model realizing in arbitrary dimension D the deformed Maxwell algebras and look for the physical meaning of the additional coordinates (f_{ab}, θ^{ab}). Using the techniques of non-linear realization, (see e.g. [15]), we generalize the results described in sect 2 for the standard Maxwell algebra [14] to those for the deformed Maxwell algebra with $(k \neq 0, b = 0)$. In such a way we obtain the generalization to AdS(dS) space-time of the model describing the particle interacting with constant values of electromagnetic field via the Lorentz force.

4.1 Standard parametrization of the coset

We consider a coset G/H with $G = \{P_a, M_{ab}, Z_{ab}\}, H = \{M_{ab}\}$ and parametrize the group element g using (x^a, θ^{ab}), the group parameters associated to the generators (P_a, Z_{ab}). Following to (2.1) we define

$$g = e^{iP_a x^a} e^{\frac{i}{2} Z_{ab} \theta^{ab}}. \tag{4.1}$$

The space-time symmetry is k-deformed Maxwell algebra (3.7) and $\{P_a, Z_{ab}\}$ form subalgebra generators isomorphic to those of AdS (dS) for $k > 0, (k < 0)$. The MC form for this coset is

$$\Omega = -ig^{-1}dg = L_P^a P_a + \frac{1}{2} L_Z^{ab} Z_{ab} + \frac{1}{2} L_M^{ab} M_{ab}, \tag{4.2}$$

where

$$L_P^a = e^b \, \Lambda_b{}^a, \quad L_Z^{cd} = -\frac{1}{k} \Lambda^{-1c}{}_a \left(\omega^{ab} - (\Lambda \, d\Lambda^{-1})^{ab}\right) \Lambda_b{}^d, \quad L_M^{cd} = 0, \tag{4.3}$$

and Λ is a vector Lorentz transformations (Lorentz harmonics) in terms of new tensorial coordinates θ^{ab} as follows

$$\Lambda_a{}^b = (e^{-k\theta})_a{}^b = \delta_a{}^b + (-k\theta)_a{}^b + \frac{1}{2!}(-k\theta)_a{}^c(-k\theta)_c{}^b + \dots . \tag{4.4}$$

Remember the indices $a, b, \dots$ are rised and lowered using the Lorentz metric $h_{ab} = (- : + \dots +)$. One-forms (e^a, ω^{ab}) are

$$\begin{aligned} e^a &= dx^c e_c{}^a = dx^c \left(\delta_c{}^a + \left(\frac{\sin(\sqrt{kr^2})}{\sqrt{kr^2}} - 1 \right) \left(\delta_c{}^a - \frac{x_c x^a}{x^2} \right) \right), \\ \omega^{ab} &= dx^c \omega_c{}^{ab} = dx^c \, \frac{\delta_c{}^{[a} \, x^{b]}}{x^2} (\cos(\sqrt{kr^2}) - 1), \end{aligned} \tag{4.5}$$

where $r = \sqrt{-x^a x_a}$ (for dS case $k = -1/R^2 < 0$, $\frac{\sin(\sqrt{kr^2})}{\sqrt{kr^2}}$ is replaced by $\frac{\sinh(\sqrt{-kr^2})}{\sqrt{-kr^2}}$ correspondingly). They satisfy the known AdS MC equations

$$de^a + \omega^a{}_b e^b = 0, \qquad d\omega^{ab} + \omega^a{}_c \omega^{cb} = -k \, e^a e^b \tag{4.6}$$

and L's in (4.3) satisfy the MC equations (3.6), with $b = 0$. Remembering $L_M^{ab} = 0$ they are

$$dL_P^a = k \, L_Z^{ac} L_{Pc}, \qquad dL_Z^{ab} = L_P^a L_P^b + k \, L_Z^{ac} L_{Zc}{}^b. \tag{4.7}$$

The particle action generalizing (2.7) for $k \neq 0$ looks as follows

$$\mathcal{L} \, d\tau = -m\sqrt{-\eta_{ab} \, L_P^{a*} L_P^{b*}} + \frac{1}{2} f_{ab} L_Z^{ab*} = -m\sqrt{-g_{ab}(x) \, \dot{x}^a \dot{x}^b} \, d\tau + \hat{A}^*, \tag{4.8}$$

where g_{ab} is the metric, now depending only on x,

$$g_{ab} = e_a{}^c e_b{}^d \eta_{cd} = \eta_{ab} + \left[\left(\frac{\sin(\sqrt{kr^2})}{\sqrt{kr^2}} \right)^2 - 1 \right] \left(\eta_{ab} - \frac{x_a x_b}{x^2} \right). \tag{4.9}$$

We obtain the metric of AdS (dS) space with radius R, where $k = 1/R^2$. The pullback $\hat{A}^*$ in (4.8) takes the explicit form

$$\begin{aligned} \hat{A}^* &= -\frac{1}{2} f_{ab} L_Z^{ba*} = -tr\left(\frac{1}{2} f \, L_Z^*\right) = +\frac{1}{2k} \, tr\left[f \, \Lambda^{-1} \left(\omega_\tau - \Lambda \, \partial_\tau \Lambda^{-1}\right) \Lambda\right] \, d\tau, \\ \omega_\tau^{cd} &= \frac{\dot{x}^{[c} x^{d]}}{x^2} \, (\cos(\sqrt{kr^2}) - 1). \end{aligned} \tag{4.10}$$

In a limit $k \to 0$ (equivalently $R \to \infty$) we obtain the undeformed Maxwell case (2.11)[14].

Now we shall describe the equations of motion following from the lagrangian (4.8). Taking the variation with respect to f_{ab} we get (we suppress the tnsor indices)

$$\omega_\tau - \Lambda\, \partial_\tau \Lambda^{-1} = 0. \tag{4.11}$$

In the limit $k \to 0$ the terms linear in k reproduce the equation (2.9). Comparing with (4.3) we see that the pullback of L_Z^{ab} to the world line vanishes on shell. The variation with respect to θ^{ab} is simplified using (4.11) and becomes the same equation as in the Maxwell case (2.8)

$$\dot{f}_{ab} = 0. \tag{4.12}$$

Finally variation with respect to x^a gives, after using (4.11), the generalization of equation of motion (2.10) describing particle moving under the Lorentz force,

$$m\, \nabla_\tau \dot{x}_a = F_{ab} \dot{x}^b, \tag{4.13}$$

where

$$\begin{aligned} \nabla_\tau \dot{x}_a &\equiv \frac{g_{ab}}{\sqrt{-g_{ef}\dot{x}^e\dot{x}^f}} \left(\sqrt{-g_{ef}\dot{x}^e\dot{x}^f}\, \partial_\tau \frac{\dot{x}^b}{\sqrt{-g_{ef}\dot{x}^e\dot{x}^f}} + \Gamma^b{}_{cd}\dot{x}^c\dot{x}^d \right), \\ \Gamma^b{}_{cd} &= \frac{1}{2} g^{ba}(g_{ac,d} + g_{ad,c} - g_{cd,a}), \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} F_{ab}(x,\theta) &= (\Lambda\, f\, \Lambda^{-1})_{cd}\, e_a{}^c\, e_b{}^d \\ &= \tilde{f}_{ab} \left(\frac{\sin(\sqrt{kr^2})}{\sqrt{kr^2}} \right)^2 - \frac{\tilde{f}_{[ac} x^c x_{b]}}{x^2} \frac{\sin(\sqrt{kr^2})}{\sqrt{kr^2}} \left(\frac{\sin(\sqrt{kr^2})}{\sqrt{kr^2}} - 1 \right) \end{aligned} \tag{4.15}$$

provided that

$$\tilde{f}_{ab} = (\Lambda\, f \Lambda^{-1})_{ab}. \tag{4.16}$$

We see that for $k \neq 0$ the generalized Lorentz force depends on θ^{cd} but in the limit $k \to 0$ we get $F_{ab} = f_{ab}$ as expected.

The interaction term $\hat{A}^*$ in the lagrangian (4.8) defines an analogue of the EM potential $\hat{A}$ as one-form in the extended bosonic space of $y^A = (x^a, \theta^{ab})$. Due to the MC eq. (4.7) its field strength is

$$\hat{F} = d\,\hat{A} = \frac{1}{2} f_{ab}\, L_P^a \wedge L_P^b + \frac{k}{2}\, f_{ab}\, L_{Zc}^a \wedge L_Z^{cb} + \frac{1}{2}\,(df_{ab}) \wedge L_Z^{ab} \tag{4.17}$$

The first term depends only on the coordinate differential forms dx^a and it can be shown from (4.15) that

$$\frac{1}{2} f_{ab} L_P^a \wedge L_P^b = \frac{1}{2}\, F_{ab}\, dx^a \wedge dx^b, \tag{4.18}$$

where F_{ab} is given in (4.15) as appears in the equation of motion (4.13). The second and third terms of (4.17) contain L_Z and df_{ab} whose pullback vanish as the result of the equations of motion (4.11) and (4.12). We can say that field strength occurring in (4.13) can be regarded as the one described by the generalized U(1) gauge potential[9] $\hat{A}$ in the extended bosonic space $(x^a, \theta^{ab}, f_{ab})$. We see that on-shell (i.e. modulo equations of motion) this field strength has only constant components f_{ab} with respect to the two-form base $L^a \wedge L^b$. We conclude that on-shell f_{ab} is a constant (see (4.12)) and the variables $(x^a(\tau), \theta^{ab}(\tau))$ satisfy the set of nonlinear differential equations, (4.11) and (4.13). If we express $\theta^{ab}(\tau)$ by using (4.11) in terms of variables $x^a(\tau)$, and substitute into $\Lambda(\theta)$ defining F_{ab}, we obtain the generalized Lorentz force, which is *nonlocal* in the variable $x^a(\tau)$.

4.2 Second parametrization of the coset

In this subsection we shall consider a different choice of the coset parametrization,

$$g' = e^{iP_a x^a} e^{\frac{i}{2} Z_{ab}\theta^{ab}} h, \qquad h = e^{\frac{i}{2} k M_{ab}\theta^{ab}} \in H. \tag{4.19}$$

The g' and g differ by an element of H and are equivalent representatives of the coset element of G/H. In particular in the $k \to 0$ limit both g and g' coincide. Using the basis (3.8) we get

$$g' = e^{i\sqrt{|k|}\mathcal{P}_a x^a} e^{\frac{i}{2} k \mathcal{J}_{ab}\theta^{ab}}. \tag{4.20}$$

The MC one-forms obtained from (4.20) can be expressed in two bases of k-deformed Maxwell algebra, (3.7) and (3.9), as follows;

$$\begin{aligned} \Omega' = -i{g'}^{-1}dg' &= L^a_{\mathcal{P}}\mathcal{P}_a + \frac{1}{2}L^{ab}_{\mathcal{M}}\mathcal{M}_{ab} + \frac{1}{2}L^{ab}_{\mathcal{J}}\mathcal{J}_{ab} \\ &= L'^a_P P_a + \frac{1}{2}L'^{ab}_Z Z_{ab} + \frac{1}{2}L'^{ab}_M M_{ab}. \end{aligned} \tag{4.21}$$

The explicit forms of the MC one-forms are

$$L^a_{\mathcal{P}} = \sqrt{|k|}\, e^a, \qquad L^{ab}_{\mathcal{M}} = \omega^{ab}, \qquad L^{ab}_{\mathcal{J}} = (\Lambda\, d\,\Lambda^{-1})^{ab}. \tag{4.22}$$

then

$$\begin{aligned} L'^a_P &= \frac{L^a_{\mathcal{P}}}{\sqrt{|k|}} = e^a, \quad L'^{ab}_M = L^{ab}_{\mathcal{J}} = (\Lambda\, d\,\Lambda^{-1})^{ab}, \\ L'^{ab}_Z &= \frac{1}{k}(L^{ab}_{\mathcal{J}} - L^{ab}_{\mathcal{M}}) = -\frac{1}{k}(\omega^{ab} - (\Lambda\, d\,\Lambda^{-1})^{ab})). \end{aligned} \tag{4.23}$$

Note that $L^a_{\mathcal{P}}$ and $L^{ab}_{\mathcal{M}}$ are given by the vielbein and the spin connection of the AdS (dS) space and $L^{cd}_{\mathcal{J}}$ is the spin connection of the "external" Lorentz space. We can interpret L'^{ab}_Z as the difference of these spin connections.

The particle action on the coset (4.19) invariant under the deformed Maxwell algebra can be obtained by replacing L by L' in the action (4.8). We get

$$\mathcal{L}'\, d\tau = -m\sqrt{-\eta_{ab}\, L'^{a*}_P L'^{b*}_P} + \frac{1}{2}f'_{ab}L'^{ab*}_Z = -m\sqrt{-g'_{ab}(x)\,\dot{x}^a\dot{x}^b}\, d\tau + \hat{A}'^*, \tag{4.24}$$

[9] The U(1) gauge transformation is considered in the extended space as $\hat{A} \to \hat{A} + d\Lambda(x, \theta, f)$ under which $\hat{F}$, therefore F_{ab}, remains invariant on-shell.

where g'_{ab} is same AdS(dS) metric (4.9) obtained in the previous parametrization,

$$g'_{ab}(x) = e_a{}^c e_b{}^d \eta_{cd} = g_{ab}(x). \tag{4.25}$$

The interaction term $\hat{A}'^*$ is written in terms of an auxiliary dynamical variable f'_{ab} as

$$\hat{A}'^* = -\frac{1}{2} f'_{ab} L'^{ba*}_Z = -\frac{1}{2} tr(f' L_Z{}') = \frac{1}{2k} tr\left[f'(\omega_\tau - \Lambda\,\partial_\tau \Lambda^{-1})\right] d\tau. \tag{4.26}$$

Then lagrangian $\mathcal{L}$ in (4.8) and $\mathcal{L}'$ in (4.24) can be identified if

$$f' = \Lambda\, f\, \Lambda^{-1}. \tag{4.27}$$

Since this is a point transformation of the coordinates, from $\{x^a, \theta^{ab}, f_{ab}\}$ to $\{x^a, \theta^{ab}, f'_{ab}\}$, the Euler-Lagrange equations of these lagrangians are equivalent.

Let us calculate the equations of motion which follow from the lagrangian (4.24). Taking the variation with respect to f'_{ab} we get

$$\omega_\tau - \Lambda \partial_\tau \Lambda^{-1} = 0, \tag{4.28}$$

It coincides with (4.11) and means that the pullback of L'^{ab}_Z to the world line vanishes on shell. In geometrical terms the "gravitational" AdS spin connection coincides on shell with the "external " Lorentz spin connection. Using (4.28) the variation of the lagrangian with respect to θ^{ab} is written as

$$\partial_\tau f'_{ab} + (\Lambda \partial_\tau \Lambda^{-1})_{[a}{}^c f'_{cb]} = \partial_\tau f'_{ab} + \omega_{\tau [a}{}^c f'_{cb]} \equiv D_\tau f'_{ab} = 0. \tag{4.29}$$

If we use the relation (4.27) it gives the same equation as (4.12) obtained in the first parametrization. Finally equation of motion for x which define the generalized non-local Lorentz force is

$$m\, \nabla_\tau \dot{x}_a = F'_{ab} \dot{x}^b, \tag{4.30}$$

where

$$F'_{ab} = f'_{cd}\, e_a{}^c\, e_b{}^d. \tag{4.31}$$

Again using the relation (4.27) we obtain

$$f' = \tilde{f} = \Lambda\, f\, \Lambda^{-1} \qquad \text{and} \qquad F'_{ab} = F_{ab} \tag{4.32}$$

or equivalently

$$\hat{F}' = \frac{1}{2} F'_{ab} dx^a \wedge dx^b = \frac{1}{2} f'_{ab}\, L'^a_P \wedge L'^b_P = \frac{1}{2} f_{ab}\, L^a_P \wedge L^b_P = \hat{F}. \tag{4.33}$$

We conclude that on-shell (i.e. modulo equations of motion) the field strength has constant components f_{ab} with respect to the 2-form base $L^a_P \wedge L^b_P$ but in the new base $L'^a_P \wedge L'^b_P$ the variables f'_{ab} are covariantly constant (see eq.(4.29)).

It will be interesting to have a physical interpretation of the non-local Lorentz force in AdS(dS) space (see (4.13) and (4.30)).

5 b-deformed Maxwell algebra in $D = 2 + 1$ and Goldstone-Nambu vector fields

In D=2+1 one can introduce second deformation parameter b. In this section after the calculation of the MC one-forms for the b-deformed coset space we shall use the resulting geometry to introduce an action for D=2+1 Abelian Goldstone-Nambu fields.

5.1 MC forms for b-deformed Maxwell algebra in $D = 2 + 1$

We consider the coset (4.1) in D=2+1 in order to define the extended space-time (x^a, θ_a) for the algebra (3.11) with $k = 0$

$$g = e^{ix^a P_a}\, e^{i\theta^a Z_a} = g_0\, e^{i\theta^a Z_a}, \qquad g_0 = e^{ix^a P_a}. \tag{5.1}$$

We compute the MC one-form in two steps,

$$\Omega = -ig^{-1}dg = e^{-i\theta^a Z_a}\, \Omega_0\, e^{i\theta^a Z_a} - i\, e^{-i\theta^a Z_a}\, d\, e^{i\theta^a Z_a}. \tag{5.2}$$

Firstly we calculate

$$\Omega_0 = g_0^{-1} dg_0 = L^a_{0P} P_a + L^a_{0Z} Z_a + L^a_{0M} M_a, \tag{5.3}$$

where

$$\begin{pmatrix} L^a_{0P} \\ L^a_{0M} \\ L^a_{0Z} \end{pmatrix} = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \delta^a{}_c + \begin{pmatrix} F_0(Y) - 1 \\ bx^2\, F_2(Y) \\ -b(x^2)^2\, F_4(Y) \end{pmatrix} O^a{}_c + \begin{pmatrix} -bx^2\, F_3(Y) \\ -b^2(x^2)^2\, F_5(Y) \\ F_1(Y) \end{pmatrix} \epsilon^a{}_{cb} x^b \right] dx^c. \tag{5.4}$$

Here $O_a{}^b = (\delta_a{}^b - \frac{x_a x^b}{x^2})$ and

$$F_i(Y) = \sum_{n=0} \frac{Y^{6n}}{(6n+i+1)!}, \qquad Y = b^{\frac{1}{3}}(x^2)^{\frac{1}{2}}, \qquad (i = 0, 1, 2, 3, 4, 5). \tag{5.5}$$

The explicit forms of functions $F_i(Y)$'s are given in appendix B.

The complete MC one-form Ω becomes

$$\Omega = L^a_P P_a + L^a_Z Z_a + L^a_M M_a, \tag{5.6}$$

with

$$\begin{aligned} \begin{pmatrix} L^a_P \\ L^a_M \\ L^a_Z \end{pmatrix} &= \left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \delta^a{}_c + \begin{pmatrix} -b^3(\theta^2)^2\, F_4(Y') \\ -b^2\theta^2\, F_2(Y') \\ F_0(Y') - 1 \end{pmatrix} \tilde{O}^a{}_c + \begin{pmatrix} -b\, F_1(Y') \\ -b^4(\theta^2)^2\, F_5(Y') \\ b^2\theta^2\, F_3(Y') \end{pmatrix} \epsilon^a{}_{cb}\theta^b \right] d\theta^c \\ &\quad + \left[\big(I_3\big)\, \delta^a{}_c + \big(V_O\big)\, \tilde{O}^a{}_c + \big(V_E\big)\, \epsilon^a{}_{cb}\theta^b \right] \begin{pmatrix} L^c_{0P} \\ L^c_{0M} \\ L^c_{0Z} \end{pmatrix}, \end{aligned} \tag{5.7}$$

where $\tilde{O}_a{}^b = (\delta_a{}^b - \frac{\theta_a\theta^b}{\theta^2})$ and L_0^c's are given in (5.4). I_3, V_O, and V_E are 3×3 matrices acting on the three vector $\Big(L_{0P}, L_{0M}, L_{0Z}\Big)$. I_3 is the unit matrix and

$$\begin{aligned}
\Big(V_O\Big) &= \begin{pmatrix} f_0(Y')-1, & b\theta^2\, f_2(Y'), & -b^3(\theta^2)^2\, f_4(Y') \\ b^3(\theta^2)^2\, f_4(Y') & f_0(Y')-1, & -b^2\theta^2\, f_2(Y') \\ -b\theta^2\, f_2(Y'), & -b^2(\theta^2)^2,\, f_4(Y') & f_0(Y')-1 \end{pmatrix}, \\
\Big(V_E\Big) &= \begin{pmatrix} b^2\theta^2\, f_3(Y'), & b^3(\theta^2)^2\, f_5(Y'), & -b\, f_1(Y') \\ b\, f_1(Y'), & b^2\theta^2\, f_3(Y'), & -b^4(\theta^2)^2\, f_5(Y') \\ -b^3(\theta^2)^2\, f_5(Y'), & -\, f_1(Y'), & b^2\theta^2\, f_3(Y') \end{pmatrix}.
\end{aligned} \tag{5.8}$$

Here the functions $F_i(Y')$'s are given in (5.5) and the functions $f_i(Y')$'s are

$$f_i(Y') \equiv \sum_{n=0} \frac{Y'^{6n}}{(6n+i)!}, \qquad Y' = b^{\frac{2}{3}}(\theta^2)^{\frac{1}{2}}, \qquad (i=0,1,2,3,4,5). \tag{5.9}$$

The explicit forms of (5.9) are listed in appendix B.

For small deformation parameter b the MC one-forms are, up to b^2,

$$\begin{aligned}
L_P^a &= dx^a - b\left((\frac{x^2}{4!}\,\epsilon^a{}_{bc}x^c + \frac{1}{2}\,\epsilon^a{}_{cd}\theta^d\,\epsilon^c{}_{be}x^e\,)dx^b + \frac{1}{2}\,\epsilon^a{}_{bc}\theta^c\,d\theta^b \right) \\
&\quad + b^2\left(\frac{x^2\,\theta^2}{2!3!}\,\tilde{O}^a{}_c\,O^c{}_b + \frac{\theta^2}{3!}\,\epsilon^a{}_{bc}\theta^c + \frac{(x^2)^2}{5!}\,\epsilon^a{}_{cd}\theta^d\,O^c{}_b + \frac{(x^2)^3}{7!}\,O^a{}_b\right)\,dx^b + \mathcal{O}(b^3), \\
L_M^a &= b\left(\frac{x^2}{3!}\,O^a{}_c\,dx^c + \epsilon^a{}_{cb}\theta^b\,dx^c\right) \\
&\quad - b^2\left(\left(\frac{\theta^2}{2!2!}\,\tilde{O}^a{}_c\,\epsilon^c{}_{bd}x^d + \frac{x^2}{4!}\,\epsilon^a{}_{cd}\theta^d\,\epsilon^c{}_{be}x^e + \frac{(x^2)^2}{6!}\,\epsilon^a{}_{bc}x^c\right)dx^b + \frac{\theta^2}{3!}\,\tilde{O}^a{}_b\,d\theta^b\right) + \mathcal{O}(b^3), \\
L_Z^a &= d\theta^a + \frac{1}{2}\epsilon^a{}_{cb}x^b\,dx^c - b\left(\frac{(x^2)^2}{5!}\,O^a{}_c + \frac{\theta^2}{2!}\,\tilde{O}^a{}_c + \frac{x^2}{3!}\,\epsilon^a{}_{db}\theta^b\,O^d{}_c\right)\,dx^c \\
&\quad + b^2\left(\left\{\frac{x^2\,\theta^2}{2!4!}\,\tilde{O}^a{}_c\,\epsilon^c{}_{bd}x^d + \left(\frac{(x^2)^2}{6!} + \frac{\theta^2}{2!3!}\right)\,\epsilon^a{}_{cd}\theta^d\,\epsilon^c{}_{be}x^e + \frac{(x^2)^3}{8!}\,\epsilon^a{}_{bc}x^c\right\}dx^b \right. \\
&\qquad\qquad \left. + \frac{\theta^2}{4!}\,\epsilon^a{}_{bc}\theta^c\,d\theta^b\right) + \mathcal{O}(b^3).
\end{aligned} \tag{5.10}$$

Using the formula (5.10) one can calculate the metric in the extended space-time $y^A = (x^a, \theta^{ab})$ with the following decomposition

$$g_{AB}(y)\dot{y}^A\dot{y}^B = g_{ab}(x,\theta)\dot{x}^a\dot{x}^b + 2\,g_{a\bar{b}}(x,\theta)\dot{x}^a\dot{\theta}^{\bar{b}} + g_{\bar{a}\bar{b}}(\theta)\dot{\theta}^{\bar{a}}\dot{\theta}^{\bar{b}}, \tag{5.11}$$

where

$$
\begin{aligned}
g_{ab}(x,\theta) &= \eta_{ab} - b\left((x\theta)\eta_{ab} - \frac{1}{2}(x_a\theta_b + x_b\theta_a)\right) \\
&\quad + b^2\left(\left(\frac{2}{7!} - \frac{1}{(4!)^2}\right)(x^2)^3\left(\eta_{ab} - \frac{x_a x_b}{x^2}\right) - \frac{x^2}{5!}(x_a\,\epsilon_{bcd}x^c\,\theta^d + x_b\,\epsilon_{acd}x^c\,\theta^d)\right. \\
&\quad \left. + \left(\frac{1}{4}(x\theta)^2 + \frac{1}{6}\,x^2\,\theta^2\right)\eta_{ab} + \frac{1}{12}\,x^2\,\theta_a\theta_b - \frac{1}{6}\,(x\theta)\,(x_a\theta_b + x_b\theta_a) - \frac{1}{6}\,\theta^2\,(x_a x_b)\right) \\
&\quad + \mathcal{O}(b^3) \\
g_{a\bar{b}}(x,\theta) &= -\frac{b}{2}\,\epsilon_{a\bar{b}c}\theta^c + \frac{b^2}{4}\left(\frac{x^2}{12}\,(\theta_a\,x_{\bar{b}} - (x\theta)\eta_{a\bar{b}}) + \theta^2\,\epsilon_{a\bar{b}c}x^c\; + \theta_{\bar{b}}\,\epsilon_{acd}x^c\theta^d\right) + \mathcal{O}(b^3), \\
g_{\bar{a}\bar{b}}(\theta) &= -b^2\,\frac{\theta^2}{4}\left(\eta_{\bar{a}\bar{b}} - \frac{\theta_{\bar{a}}\theta_{\bar{b}}}{\theta^2}\right) + \mathcal{O}(b^3).
\end{aligned}
\tag{5.12}
$$

It can be checked from the general formula (5.7) for L_P^a that in all orders of b the metric $g_{\bar{a}\bar{b}}$ does not depend on the space-time coordinates x^a.

5.2 Nonlinear action for $D = 2+1$ Goldstone-Nambu vector fields

In order to introduce the D=2+1 Goldstone-Nambu fields $\theta^a(x)$ we replace the coset (5.1) describing the coordinates (x^a, θ^a) in our generalized D=2+1 space-time by

$$
\tilde{g} = e^{iP_a x^a} e^{iZ_a \theta^a(x)}. \tag{5.13}
$$

Here the independent coordinates are x^a, and the fields $\theta^a(x)$ describe three-dimensional submanifold in (x^a, θ^a). The fields $\theta^a(x)$ transform homogeneously under the so(2,1) rotations generated by M_a, but inhomogenously under the generators Z_a, what implies spontaneous breaking of Z_a symmetries. The Goldstone-Nambu fields describing spontaneously broken directions in extended space-time were introduced by nonlinear realization method [27, 28] in supersymmetric theories. The broken directions were provided by the odd superspace degrees of freedom describing fermionic Goldstino fields [19], or by introducing in D dimensional space-time the p-brane fields ($D > p+1$) (spontaneously broken directions are transversal to the p-brane, see for example [16, 29, 30]). In this section we shall convert in $D = 2+1$ the additional degrees of freedom θ^a into Abelian vectorial Goldstone-Nambu fields $\theta^a(x)$ which can be also interpreted as describing a 3-brane in $D = 6$ space time (x^a, θ^a).

In order to study the dynamics of fields $\theta^a(x)$ we should calculate, using (5.13), the left-invariant MC one-forms

$$
\tilde{\Omega} = -i\tilde{g}^{-1}d\tilde{g} = P_a\;\tilde{e}^a + \frac{1}{2}Z_{ab}\;\tilde{\omega}^{ab} + \frac{1}{2}M_{ab}\;\tilde{l}^{ab}, \tag{5.14}
$$

where the only independent differentials are dx^a. The one-form $\tilde{\Omega}$ can be obtained from Ω in (5.2) by taking the pullback with respect to $x^a \to \theta^a$, then

$$
d\theta^{a\star} = \frac{\partial\theta^a(x)}{\partial x^b}dx^b; \tag{5.15}
$$

in such a way every form is defined on x^a-space. One can employ further the one-forms (5.2) with $k=0, b\neq 0$, explicitly calculated in section 5.1. From (5.7) we obtain

$$\tilde{e}^a = L_P^{a\star} = \left(e^a{}_b(x,\theta(x)) + f^a{}_c(\theta(x))\frac{\partial\theta^c(x)}{\partial x^b}\right) dx^b \equiv \tilde{e}^a{}_b(x,\theta(x))dx^b. \tag{5.16}$$

where

$$\begin{aligned} e^a{}_b(x,\theta) &= \delta^a{}_b - b\left(\frac{x^2}{4!}\,\epsilon^a{}_{bc}x^c + \frac{1}{2}\left((x^c\theta_c)\delta^a{}_b - x^a\theta_b\right)\right) \\ &\quad + b^2\left(\left(\left(\frac{x^2\,\theta^2}{2!3!} + \frac{(x^2)^3}{7!}\right)\delta^a{}_b - \left(\frac{\theta^2}{2!3!} + \frac{(x^2)^2}{7!}\right)x^a x_b - \frac{x^2}{2!3!}\theta^a\theta_b + \frac{(x\theta)}{2!3!}\theta^a x_b\right.\right. \\ &\quad \left.\left. + \left(\frac{\theta^2}{3!} + \frac{(x^2)^2}{5!}\right)\epsilon^a{}_{bc}\theta^c - \frac{(x^2)}{5!}\,\epsilon^a{}_{cd}\theta^d\,x^c x_b\right) + \mathcal{O}(b^3),\right. \end{aligned} \tag{5.17}$$

and

$$f^a{}_c(\theta(x)) = -\frac{b}{2}\,\epsilon^a{}_{cb}\theta^b + \mathcal{O}(b^3). \tag{5.18}$$

One can check that $f^a{}_c(\theta)$ depends only on θ^a and the dreibein $\tilde{e}^a{}_b$ is linear in the derivatives of the Goldstone fields.

In order to construct the action which is invariant under the b-deformed Maxwell group one can use the Volkov-Akulov formula for invariant D=2+1 action,

$$S = \int (-\frac{1}{3!})\epsilon_{abc}\, L_P^{a\star} L_P^{b\star} L_P^{c\star} = \int d^3x\,\mathcal{L}_\theta, \qquad \mathcal{L}_\theta = \det(\tilde{e}^a{}_b). \tag{5.19}$$

Using (5.16)-(5.18) one can write explicitly the terms up to b^2,

$$\tilde{e}^a{}_b(x,\theta) = e^a{}_b(x,\theta) + b\,h^a{}_b, \qquad h^a{}_b = -\frac{1}{2}\,\epsilon^a{}_{cd}\theta^d\,\frac{\partial\theta^c(x)}{\partial x^b}. \tag{5.20}$$

Using

$$\begin{aligned} \det(\tilde{e}^a{}_b) &= -\frac{1}{3!}\epsilon_{abc}\epsilon^{def}\,\tilde{e}^a{}_d\,\tilde{e}^b{}_e\,\tilde{e}^c{}_f \\ &= \det(e^a{}_b)\left(1 + b\,(e^{-1})^b{}_a\,h^a{}_b + \frac{b^2}{2}\,(e^{-1})^a{}_c\,(e^{-1})^b{}_d\,h^c{}_{[a}h^d{}_{b]}\right) + \mathcal{O}(b^3), \\ \det(e^a{}_b) &= 1 - b\,(x\theta) + b^2\left(-\frac{3(x^2)^3}{2240} + \frac{x^2\,\theta^2}{12} + \frac{(x\theta)^2}{3}\right) + \mathcal{O}(b^3), \\ (e^{-1})^b{}_a &= \delta^a{}_b + b\left(\frac{x^2}{4!}\,\epsilon^a{}_{bc}x^c + \frac{1}{2}\left((x^c\theta_c)\delta^a{}_b - x^a\theta_b\right)\right) + \mathcal{O}(b^2) \end{aligned} \tag{5.21}$$

we obtain

$$\begin{aligned} \mathcal{L}_\theta &= \det(\tilde{e}^a{}_b) = 1 - b\left((x\theta) + \frac{1}{2}\epsilon_{abc}\theta^a\frac{\partial\theta^c}{\partial x_b}\right) + b^2\left(\left(-\frac{3(x^2)^3}{2240} + \frac{x^2\,\theta^2}{12} + \frac{(x\theta)^2}{3}\right)\right. \\ &\quad \left. - \frac{x^2}{48}((x\theta)\delta_j{}^i - x_j\theta^i)\frac{\partial\theta^j}{\partial x^i} + \frac{(x\theta)}{4}\epsilon_{abc}\theta^a\frac{\partial\theta^c}{\partial x_b} + \frac{1}{8}\,\epsilon_{abc}\epsilon^{def}\theta^a\theta_d\frac{\partial\theta^b}{\partial x^e}\frac{\partial\theta^c}{\partial x^f}\right) + \mathcal{O}(b^3). \end{aligned} \tag{5.22}$$

The lagrangian density (5.22) contains as one of two terms linear in b the exact topological lagrangian for D=2+1 Chern-Simons field

$$\mathcal{L}^{CS} = -\frac{b}{2}\,\epsilon_{abc}\theta^a \frac{\partial \theta^b}{\partial x_c}. \tag{5.23}$$

If we consider higher order terms in b they can be treated as describing new interaction vertices and the Nambu-Goldstone field $\theta^a(x)$ looses its topological nature. The appearance of the terms depending explicitly on x^a and θ^a in (5.22) is related with the curved geometry in the extended space (see (5.12)). Although the explicit formula (5.22) looks complicated the covariance of the action (5.19) under the deformed Maxwell group transformations which describe the group of motions in the curved space (x^a, θ^a) follows from our construction obtained by using the nonlinear realization techniques.

6 Outlook

In this paper we consider deformations of the Maxwell algebra. The general mathematical techniques permit us to solve the problem of complete classification of these deformations. The commuting generators Z_{ab} in (1.1) are becoming non-abelian in arbitrary dimension D and are promoted to the $\frac{D(D-1)}{2}$ generators of the $so(D-1,1)$ Lorentz algebra. The particle dynamics in the $\frac{D(D+1)}{2}$ dimensional coset (4.1) becomes the theory of point particles moving on AdS (for $k>0$) or dS (for $k<0$) group manifolds in external electromagnetic fields. If we use standard formula (4.8) for the particle action in curved space-time one can show that the particle moves only in the space-time sector $(x^a, \theta^{ab}=0)$ of the extended space-time (x^a, θ^{ab}) with a non-local Lorentz force. The supplementary coordinates θ^{ab} generated by Z_{ab}, enter only in MC one-forms and in particular they will appear in the model only in the term representing the electromagnetic coupling. It is a result of the field equations that the components of electromagnetic field strength defined in the basis of momenta one-forms L^a are constant on-shell(see (4.18)).

In "exotic" dimension $D = 2+1$ the symmetry corresponding to the two parameter deformation of Maxwell algebra is less transparent. The coset (4.1) in D=2+1 if $b \neq 0$ is neither the group manifold nor even the symmetric coset space. In order to find the dynamical realization of deformed Maxwell algebra with $b \neq 0$ in D=2+1 space-time, in section 5 we consider the D=2+1 field theoretical model obtained by the assumption that the coordinates x^a are primary and the coordinates θ^a describe the Goldstone field values. We obtained a non-linear lagrangian for vector Goldstone field containing the bi-linear kinetic term describing exactly the D=2+1 CS Abelian action.

We would like to point out some problems which deserve still further consideration.

1) The Maxwell algebra was obtained as a deformation of the relativistic Poincare algebra in the presence of constant electromagnetic background. One can observe that the relation (1.1) is dual (in the sense of Fourier transformation) to the canonical non-commutativity of the Minkowski space-time (see e.g. [31, 32])

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu} \tag{6.1}$$

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which, as it is argued [33], describes in algebraic approximation of the quantum gravity background. It could be interesting to study this parallelism further.

2) Maxwell algebra contains in four dimensions three quadratic and one quartic Casimirs [1, 2] (see (2.19)). In arbitrary dimension D the Casimir $C_2 = Z_{ab}Z^{ab}$ can be incorporated in the particle action by means of the following extension of the action (2.7)

$$\hat{L} = \pi_a \dot{x}^a + \frac{1}{2} f_{ab}\left(\dot{\theta}^{ab} + \frac{1}{2} x^{[b}\dot{x}^{b]}\right) - \frac{\lambda}{2}(\pi^2 + m^2) - \frac{\lambda'}{2}(f^2 + m'^2) \tag{6.2}$$

where λ, λ' are Lagrange multipliers. The action (6.2) treats symmetrically the dynamics of x^a and θ^{ab} variables. One obtains the following second order lagrangian

$$\tilde{L} = -m\sqrt{-\dot{x}^2} - m'\sqrt{-\left(\dot{\theta}^{ab} + \frac{1}{2} x^{[b}\dot{x}^{b]}\right)^2}. \tag{6.3}$$

Such a model could possibly relate the additional coordinates θ^{ab} with spin-like degrees of freedom. It should be interesting to consider the model (6.2) in detail and further extend it to the deformed Maxwell algebra geometries, using the result of section 3.

3) As we already mentioned, the deformation parameter k with the dimensionality $[L^{-2}]$ can be described by the formula $|k| = \frac{1}{R^2}$, and interpreted as the AdS(dS) radius for $k > 0 (k < 0)$. The parameter b, with the dimensionality $[L^{-3}]$, if $k = 0$ is related with the closure of the quadrilinear relation for the following non Abelian translation generators P_a,

$$[[P_a, P_b], [P_c, P_d]] = ib\,(\eta_{a[c}\epsilon_{bd]e} - \eta_{b[c}\epsilon_{ad]e})P^e. \tag{6.4}$$

This relation is an example of higher order Lie algebra for $n = 4$ [34, 35]. It is an interesting task to understand the translations (6.4) as describing some D=2+1 dimensional curved manifold.

4) Recently in [13, 14] there were considered an infinite sequential extensions of the Maxwell algebra with additional tensorial generators. The concrete form of these extensions can be determined by studying the Chevalley-Eilenberg cohomologies at degree two. The point particle models related with these Poincare algebra extensions have been studied in [13]. There appears an interesting question of the dynamical and physical interpretation of the additional tensorial degrees of freedom.

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A Determination of transformation matrix $U(b,k)$

A.1 $U^-(b,k)$ for $\det g < 0$

Here we discuss how the transformation matrix $U^-(b,k)$ in (3.21) is determined. We will see (P,M,Z) is related only to dS generators $(\mathcal{P},\mathcal{M},\mathcal{J})$ using *real component* matrix $U^-(b,k)$ for any (b,k) in the $\det g<0$ region (IV). We fix it becomes that of b-deformation for $k=0$ in (3.14),

$$U^-(b,k=0)=U_b. \tag{A.1}$$

Near the b-axis we can find $U^-(b,k)$ as the perturbation for small k. It tells a structure of the matrix as

$$U^-(b,k)=\begin{pmatrix} \frac{f_3(\kappa)+\frac{\kappa^2}{9}f_4(\kappa)}{\sqrt{3}b^{1/3}} & \frac{2\kappa}{3\sqrt{3}}(f_1(\kappa)+\frac{\kappa}{3}f_2(\kappa)) & \frac{f_1(\kappa)+\frac{\kappa}{3}f_2(\kappa)}{\sqrt{3}b^{2/3}} \\ -\frac{f_3(\kappa)-\frac{\kappa^2}{9}f_4(\kappa)}{3b^{1/3}} & \frac{2}{3}\left(1+\frac{\kappa}{3}(f_1(\kappa)-\frac{\kappa}{3}f_2(\kappa))\right) & \frac{f_1(\kappa)-\frac{\kappa}{3}f_2(\kappa)}{3b^{2/3}} \\ \frac{f_3(\kappa)-\frac{\kappa^2}{9}f_4(\kappa)}{3b^{1/3}} & \frac{1}{3}\left(1-\frac{2\kappa}{3}(f_1(\kappa)-\frac{\kappa}{3}f_2(\kappa))\right) & -\frac{f_1(\kappa)-\frac{\kappa}{3}f_2(\kappa)}{3b^{2/3}} \end{pmatrix} \tag{A.2}$$

where

$$\kappa=\frac{k}{b^{2/3}},\qquad 1-\frac{4\kappa^3}{27}>0,\qquad \text{for}\qquad \det g<0. \tag{A.3}$$

For small κ, $f_j(\kappa)$'s are polynomials of κ^3 and $f_j(0)=1$,

$$\begin{aligned} f_1(\kappa)&=\left(1+\frac{5\,\kappa^3}{3^4}+\frac{44\,\kappa^6}{3^8}+\dots\right), & f_2(\kappa)&=\left(1+\frac{7\,\kappa^3}{3^4}+\frac{65\,\kappa^6}{3^8}+\dots\right) \\ f_3(\kappa)&=\left(1+\frac{4\,\kappa^3}{3^4}+\frac{35\,\kappa^6}{3^8}+\dots\right), & f_4(\kappa)&=\left(1+\frac{8\,\kappa^3}{3^4}+\frac{77\,\kappa^6}{3^8}+\dots\right). \end{aligned} \tag{A.4}$$

General forms of $f_j(\kappa)$'s are found by requiring that the $(\mathcal{P},\mathcal{M},\mathcal{J})$ satisfy the dS algebra at point (b,k) in $\det g<0$. First from $[\mathcal{J}_a,\mathcal{J}_b]=-(-i)\epsilon_{abc}\mathcal{J}^c$ we have

$$f_3(\kappa)=\frac{(\tilde f_1(\kappa))^2}{1-\frac{2}{3}\kappa\,\tilde f_1(\kappa)}+\frac{\kappa^2}{9}f_4(\kappa),\qquad \tilde f_1(\kappa)\equiv f_1(\kappa)-\frac{\kappa}{3}f_2(\kappa). \tag{A.5}$$

and $\tilde f_1(\kappa)$ satisfies a third order equation

$$\left(1-\frac{4\kappa^3}{27}\right)(\tilde f_1(\kappa))^3+\kappa\,\tilde f_1(\kappa)-1=0. \tag{A.6}$$

Next $[\mathcal{M}_a,\mathcal{M}_b]=-(-i)\epsilon_{abc}\mathcal{M}^c$ is satisfied using above. $[\mathcal{P}_a,\mathcal{J}_b]=0$ fix $f_4(\kappa)$ as a function of $f_1(\kappa)$ and $f_2(\kappa)$ as

$$f_4(\kappa)=\frac{3\left(f_2(\kappa)\left(1-\frac{\kappa}{3}\tilde f_1(\kappa)\right)-(\tilde f_1(\kappa))^2\right)\,\tilde f_1(\kappa)}{\kappa\left(1-\frac{2\kappa}{3}\,\tilde f_1(\kappa)\right)\left(1+\frac{\kappa}{3}\,\tilde f_1(\kappa)\right)}. \tag{A.7}$$

From $[\mathcal{P}_a, \mathcal{P}_b] = +(-i)\epsilon_{abc}\mathcal{M}^c$ we determine $f_2(\kappa)$

$$f_2(\kappa) \; = \; \frac{1}{(1-\frac{4\kappa^3}{27})\, f_1(\kappa)}. \tag{A.8}$$

Using (A.6) and (A.8) we get

$$f_1(\kappa)^3 \; = \; \frac{1+\sqrt{1-\frac{4\kappa^3}{27}}}{2\sqrt{1-\frac{4\kappa^3}{27}}^{\,3}} \; . \tag{A.9}$$

$f_1(\kappa)$ is the real cubic root of this equation satisfying $f_1(0) = 1$.

In this way we have determined all functions $f_j(\kappa)$. They are shown to give generators verifying all the dS commutation relations in (3.15). In small κ expansion they agree with the perturbative expansion (A.4) around b-deformation ($k = 0$).

It is also seen that they are singular on the degenerate curve

$$-\;A(b,k) = \left(\frac{b}{2}\right)^2 - \left(\frac{k}{3}\right)^3 = \frac{b^2}{4}\left(1-\frac{4\kappa^3}{27}\right) = +0. \tag{A.10}$$

It is very interesting to see if the generators are analytically continuating to those of k^--deformation on the $b = 0, k < 0$ line, that is if

$$U^-(0,k) = U_k \tag{A.11}$$

holds for $k < 0$. It is shown by taking a limit

$$b \to 0, \quad k < 0\;(\text{fixed}); \quad \kappa = \frac{k}{(b^2)^{\frac{1}{3}}} \to -\infty. \tag{A.12}$$

In doing it

$$f_1(\kappa)^3 \quad \to \quad -\frac{27}{8\kappa^3} = \frac{27b^2}{-8k^3}, \tag{A.13}$$

then the leading terms of f_i's are

$$(f_1,\, f_2,\, f_3,\, f_4) \to \left(\frac{3\,b^{2/3}}{-2k}, \frac{9\,b^{4/3}}{2k^2}, \frac{\sqrt{3}\,b^{1/3}}{2\sqrt{-k}}, \frac{9\sqrt{3}\;b^{5/3}}{2\sqrt{-k}^{\,5}}\right). \tag{A.14}$$

Taking this limit in (A.2) it goes to U_k of k^- deformation in (3.12).

A.2 $\quad U^+(b,k)$ for $\det g > 0$

Similarly we determine the transformation matrix $U^+(b,k)$ for $\det g > 0$ in (3.20). We will see (P, M, Z) is related only to AdS generators $(\mathcal{P}, \mathcal{M}, \mathcal{J})$ using *real component* matrix $U^+(b,k)$ for any (b,k) in the $\det g > 0$ region (III). We fix it becomes that of k^+-deformation for $b = 0$ in (3.12),

$$U^+(b=0, k>0) = U_k. \tag{A.15}$$

Near the k-axis we can find $U^+(b,k)$ as the perturbation for small b. It tells a structure of the matrix as

$$U^+(b,k) = \begin{pmatrix} \frac{1}{\sqrt{k}}\,h_3(\beta) & \beta\,h_4(\beta) & \frac{3\beta}{2k}\,h_4(\beta) \\ -\frac{\beta}{\sqrt{k}}\,h_1(\beta) & -2\beta^2\,h_2(\beta) & -\frac{1}{k}\,h_5(\beta) \\ \frac{\beta}{\sqrt{k}}\,h_2(\beta) & (1+2\beta^2\,h_1(\beta)) & \frac{1}{k}\,(1+3\beta^2\,h_1(\beta)) \end{pmatrix} \tag{A.16}$$

where

$$\beta = \frac{b}{k^{3/2}}, \quad 1-\frac{27\beta^2}{4} > 0, \quad \text{for} \quad \det g > 0. \tag{A.17}$$

For small β, $h_j(\beta)$'s are polynomials of β^2 and $h_j(0)=1$,

$$\begin{aligned} h_1(\beta) &= 1 + 4\,\beta^2 + 21\,\beta^4 + \dots, \qquad & h_2(\beta) &= 1 + 5\,\beta^2 + 28\,\beta^4 + \dots \\ h_3(\beta) &= 1 + \frac{15\,\beta^2}{2^3} + \frac{1155\,\beta^4}{2^7} + \dots, & h_4(\beta) &= 1 + \frac{35\,\beta^2}{2^3} + \frac{3003\,\beta^4}{2^7} + \dots, \\ h_5(\beta) &= 1 + 3\,\beta^2 + 15\,\beta^4 + \dots. \end{aligned} \tag{A.18}$$

General forms of $h_j(\beta)$'s are found by requiring that the $(\mathcal{P},\mathcal{M},\mathcal{J})$ satisfy the AdS algebra (3.13) with $k>0$ at point (b,k) in $\det g > 0$.

The results are

$$\begin{aligned} h_1(\beta) &= \frac{h_2(\beta)\,\left(1+2\,\beta^2\,h_2(\beta)\right)}{1+3\,\beta^2\,h_2(\beta)}, \\ h_3(\beta) &= \frac{\left(2+3\,\beta^2\,h_2(\beta)\right)}{2\,\left(1+2\,\beta^2\,h_2(\beta)\right)}\sqrt{\frac{\left(1+3\,\beta^2\,h_2(\beta)\right)}{\left(1-\frac{27\,\beta^2}{4}\right)\,h_2(\beta)}}, \\ h_4(\beta) &= \sqrt{\frac{h_2(\beta)}{\left(1-\frac{27\,\beta^2}{4}\right)\,\left(1+3\,\beta^2\,h_2(\beta)\right)}}, \\ h_5(\beta) &= \frac{1+\frac{9\,\beta^2\,h_2(\beta)}{4}}{\left(1-\frac{27\,\beta^2}{4}\right)\,\left(1+3\,\beta^2\,h_2(\beta)\right)^2}, \end{aligned} \tag{A.19}$$

and $h_2(\beta)$ is determined by a third order equation

$$1-\left(1-9\,\beta^2\right)\,h_2(\beta) - 4\,\beta^2\left(1-\frac{27\,\beta^2}{4}\right)\,h_2(\beta)^2 - 4\,\beta^4\left(1-\frac{27\,\beta^2}{4}\right)\,h_2(\beta)^3 = 0, \tag{A.20}$$

whose real solution is

$$h_2(\beta) = \frac{1}{3\,\beta^2}\left(\frac{1}{\sqrt{1-\frac{27\,\beta^2}{4}}}\cos\left(\frac{1}{3}\arctan\left(\frac{3\,\sqrt{3}\,\beta}{2\,\sqrt{1-\frac{27\,\beta^2}{4}}}\right)\right)-1\right). \tag{A.21}$$

In this way we have determined all functions $h_j(\beta)$ having the small β expansion in (A.18) thus $U^+(b,k)$ becomes U_k for $b=0, k>0$. It is also seen that they are singular on the degenerate curve

$$A(b,k) = \left(\frac{k}{3}\right)^3 - \left(\frac{b}{2}\right)^2 = \left(\frac{k}{3}\right)^3\left(1-\frac{27\beta^2}{4}\right) = +0. \tag{A.22}$$

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B f_j and F_j

Here we give results of summations of functions f_j in (5.9) and F_j in (5.5). $f_j(Y) \equiv f_j(\alpha = 1, Y)$ are[10]

$$
\begin{aligned}
f_0(\alpha, Y) &= \frac{1}{3}\Bigg(2\,\cos\left(\frac{\sqrt{3}\,\alpha\,Y}{2}\right)\,\cosh\left(\frac{\alpha\,Y}{2}\right) + \cosh(\alpha\,Y)\Bigg),\\
f_1(\alpha, Y) &= \frac{1}{3Y}\Bigg(\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,\alpha\,Y}{2}\right)\,\cosh\left(\frac{\alpha\,Y}{2}\right)\\
&\qquad\qquad + \cos\left(\frac{\sqrt{3}\,\alpha\,Y}{2}\right)\,\sinh\left(\frac{\alpha\,Y}{2}\right) + \sinh(\alpha\,Y)\Bigg),\\
f_2(\alpha, Y) &= \frac{1}{3\,Y^2}\Bigg(\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,\alpha\,Y}{2}\right)\,\sinh\left(\frac{\alpha\,Y}{2}\right)\\
&\qquad\qquad - \cos\left(\frac{\sqrt{3}\,\alpha\,Y}{2}\right)\,\cosh\left(\frac{\alpha\,Y}{2}\right) + \cosh(\alpha\,Y)\Bigg),\\
f_3(\alpha, Y) &= \frac{1}{3\,Y^3}\Bigg(-2\,\cos\left(\frac{\sqrt{3}\,\alpha\,Y}{2}\right)\,\sinh\left(\frac{\alpha\,Y}{2}\right) + \sinh(\alpha\,Y)\Bigg),\\
f_4(\alpha, Y) &= \frac{1}{3\,Y^4}\Bigg(-\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,\alpha\,Y}{2}\right)\,\sinh\left(\frac{\alpha\,Y}{2}\right)\\
&\qquad\qquad - \cos\left(\frac{\sqrt{3}\,\alpha\,Y}{2}\right)\,\cosh\left(\frac{\alpha\,Y}{2}\right) + \cosh(\alpha\,Y)\Bigg),\\
f_5(\alpha, Y) &= \frac{1}{3\,Y^5}\Bigg(-\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,\alpha\,Y}{2}\right)\,\cosh\left(\frac{\alpha\,Y}{2}\right)\\
&\qquad\qquad + \cos\left(\frac{\sqrt{3}\,\alpha\,Y}{2}\right)\,\sinh\left(\frac{\alpha\,Y}{2}\right) + \sinh(\alpha\,Y)\Bigg)
\end{aligned}
\tag{B.1}
$$

and $F_j(Y)$'s are

$$
\begin{aligned}
F_0(Y) &= \frac{1}{3Y}\left(\sqrt{3}\,\cosh\left(\frac{Y}{2}\right)\,\sin\left(\frac{\sqrt{3}\,Y}{2}\right) + \cos\left(\frac{\sqrt{3}\,Y}{2}\right)\,\sinh\left(\frac{Y}{2}\right) + \sinh(Y)\right),\\
F_1(Y) &= \frac{1}{3Y^2}\left(\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,Y}{2}\right)\,\sinh\left(\frac{Y}{2}\right) - \cos\left(\frac{\sqrt{3}\,Y}{2}\right)\,\cosh\left(\frac{Y}{2}\right) + \cosh(Y)\right),\\
F_2(Y) &= \frac{1}{3\,Y^3}\left(-2\,\cos\left(\frac{\sqrt{3}\,Y}{2}\right)\,\sinh\left(\frac{Y}{2}\right) + \sinh(Y)\right),\\
F_3(Y) &= \frac{1}{3\,Y^4}\left(-\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,Y}{2}\right)\,\sinh\left(\frac{Y}{2}\right) - \cos\left(\frac{\sqrt{3}\,Y}{2}\right)\,\cosh\left(\frac{Y}{2}\right) + \cosh(Y)\right),\\
F_4(Y) &= \frac{1}{3\,Y^5}\left(-\sqrt{3}\,\cosh\left(\frac{Y}{2}\right)\,\sin\left(\frac{\sqrt{3}\,Y}{2}\right) + \cos\left(\frac{\sqrt{3}\,Y}{2}\right)\,\sinh\left(\frac{Y}{2}\right) + \sinh(Y)\right),\\
F_5(Y) &= \frac{1}{3\,Y^6}\left(-3 + 2\,\cos\left(\frac{\sqrt{3}\,Y}{2}\right)\,\cosh\left(\frac{Y}{2}\right) + \cosh(Y)\right)
\end{aligned}
\tag{B.2}
$$

[10] For x^a is timelike $Y^6 = -b^2 r^6$, $r = \sqrt{-x^2}$. In this case trigometric functions and trigonometric ones are interchanged.

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For small b we have expansions of f_j and F_j as

$$
\begin{aligned}
F_i(Y) &= \frac{1}{(i+1)!} + \frac{b^2\,(x^2)^3}{(i+7)!} + \mathcal{O}(b^4), \\
F_i(Y') &= \frac{1}{(i+1)!} + \mathcal{O}(b^4), \qquad\qquad f_i(Y') = \frac{1}{i!} + \mathcal{O}(b^4).
\end{aligned} \tag{B.3}
$$

Keeping up to b^2,

$$
\begin{pmatrix} L^a_{0P} \\ L^a_{0M} \\ L^a_{0Z} \end{pmatrix} = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \delta^a{}_c + \begin{pmatrix} b^2(x^2)^3/7! \\ bx^2/3! \\ -b(x^2)^2/5! \end{pmatrix} O^a{}_c + \begin{pmatrix} -bx^2/4! \\ -b^2(x^2)^2/6! \\ \frac{1}{2} + b^2(x^2)^3/8! \end{pmatrix} \epsilon^a{}_{cb} x^b \right] dx^c. \tag{B.4}
$$

and

$$
\begin{aligned}
\begin{pmatrix} L^a_{P} \\ L^a_{M} \\ L^a_{Z} \end{pmatrix} &= \left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \delta^a{}_c + \begin{pmatrix} 0 \\ -b^2\theta^2/3! \\ 0 \end{pmatrix} \tilde{O}^a{}_c + \begin{pmatrix} -b/2! \\ 0 \\ b^2\theta^2/4! \end{pmatrix} \epsilon^a{}_{cb} \theta^b \right] d\theta^c. \\
&\quad + \left[(I_3)\, \delta^a{}_c + \Big(V_O \Big) \tilde{O}^a{}_c + \Big(V_E \Big) \epsilon^a{}_{cb} \theta^b \right] \begin{pmatrix} L^c_{0P} \\ L^c_{0M} \\ L^c_{0Z} \end{pmatrix},
\end{aligned} \tag{B.5}
$$

where L^c_0's are given in (B.4) and

$$
\begin{aligned}
\Big(V_O \Big) &= \begin{pmatrix} 0 & b\theta^2/2! & 0 \\ 0 & 0 & -b^2\theta^2/2! \\ -b\theta^2/2! & -b^2(\theta^2)^2/4! & 0 \end{pmatrix}, \\
\Big(V_E \Big) &= \begin{pmatrix} b^2\theta^2/3! & 0 & -b \\ b & b^2\theta^2/3! & 0 \\ 0 & -1 & b^2\theta^2/3! \end{pmatrix}.
\end{aligned} \tag{B.6}
$$

Then up to b^2 we have the result of (5.10).

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